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Fixed points and unfounded chains

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Abstract

By an unfounded chain for a function $f : X \to X$ we mean a sequence $(x_n)_{n\in\omega}$ of elements of X s.t. $fx_{n+1} = x_n$ for every n. Unfounded chains can be regarded as a generalization of fixed points, but on the other hand are linked with concepts concerning non-well-founded situations, as ungrounded sentences and the hypergame. In this paper, among other things, we prove a lemma in general topology, we exhibit an extensional recursive function from the set of sentences of *PA* into itself without an unfounded chain, and we prove that every term in a Magari algebra (or diagonalizable algebra) has an unfounded chain.

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0. The problem

Let f be a function from a non-empty set X into itself. For every element x_0 of X we can inductively define the sequence $x_0,\ fx_0, f^2x_0, \dots$. If suitable conditions about the set X and the function f are satisfied, a fixed-point theorem can be proved: in this case, there is an element $a \in X$ s.t. $fa = a$.

We will consider a slightly different pattern:

Definition 1. We say that a sequence $(x_n)_{n\in\omega}$ of elements of X is an *unfounded chain* (or an *infinite descending chain*) for the function f if $fx_{n+1} = x_n$ for every n.

(As usual, ω stands for the set of natural numbers; in the sequel we will often omit "$n \in \omega$" and write briefly (x_n).)

Of course, not all functions have an unfounded chain (think of the successor function in the set ω). On the other hand, if f has a fixed point a, it has also the constant

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unfounded chain $(a)_{n\in\omega}$. More generally, if f has a periodic point (or a finite cycle, i.e. if there are $x_0, x_1, \ldots, x_n$, s.t. $fx_0 = x_1, \ldots, fx_n = x_0$), then f has an unfounded chain.

This paper is concerned with the following problem:

find conditions on the set X and the function f that guarantee the existence of an unfounded chain.

In particular, we will try to find solutions to our problem considering known fixed point theorems and weakening the hypotheses there.

As stated, the problem is very general. In fact, our main interest is in *functions from the set Σ of sentences of PA into itself*. However, at the beginning of the paper we will examine more general situations, to find tools and lemmas which will be useful later.

In Section 1 connections between the problem and known paradoxes are discussed. In Section 2, after some remarks, a result in general topology is proved (Theorems 2 and 3). From Section 3 on, we are concerned with logic. In Section 4 we exhibit an extensional recursive function from Σ into itself that admits no unfounded chain. In Section 5 we prove a theorem regarding terms in Magari algebras (or diagonalizable algebras), while Section 6 describes a typical example in the same context. Other remarks and results are discussed in Section 7.

Throughout the paper, a number of open problems are posed.

1. Self-reference vs. ungroundedness

In most classical logic paradoxes, the argument rests on self-reference and negation; in particular, self-reference is usually regarded as the element responsible for paradoxes. On the other hand, paradoxical arguments can be partially translated into formal contexts, so that they lead to important results. In this framework, self-reference is deeply connected with fixed point theorems: think of the Diagonalization Lemma in Peano arithmetic, which allows self-referential constructions and, as a consequence, yields classical incompleteness results.

But there are also paradoxes where the crucial role is played by unfounded relations instead of self-reference. In fact, to avoid all paradoxical sentences, Kripke (see [8]) introduced the concept of an *ungrounded sentence*, i.e. a sentence such that the process of determining its truth value by examining the sentences it refers to, does not terminate, either because one finds again an already considered sentence (finite cycle or, in particular, self-reference), or because there is an infinite regression. Of course, we cannot define a rank, or a level for an ungrounded sentence.

Let us briefly review three paradoxes based on an unfounded relation (see [1, 3] for further discussion).

(i) There are denumerable many people $a_0, a_1, a_2, \ldots$, and each one says the same sentence, "*at least one of the people after me* [i.e. whose index is greater than mine] *is lying*". It is not possible to assign consistent truthvalues T, F to the statements, because every T must be followed by at least one F, whilst every F must be followed only by T's.

In contrast to the liar's paradox, there is no self-reference, not even an indirect one: if a_i speaks about a_j's sentence, then a_j does not speak about a_i's sentence. A paradox also arises if all of the statements are "*Only finitely many of the people coming after me are telling the truth*"; note that in this version negation does not appear (neither does implication).

(ii) (See [18].) A game G between two players A and B is said to be *grounded* if every game of G terminates after finitely many moves (even if there is no fixed bound on the lengths of games). Define the *hypergame* as follows: player A chooses a grounded game G, then player B makes the first move in G and the game continues according to the rules of G.

Since G is grounded, also this game has an end; thus, hypergame is grounded. As a consequence, in a game of hypergame, player A can choose the hypergame itself as a grounded game, giving B the right to choose. But, if B in turn chooses hypergame, and so on, we get an infinite game in which both players move according to the rules of a grounded game.

(iii) (See [15]; this paradox dates back to 1953, while the two previous ones were introduced only few years ago.) In naive set-theory, a set A is said to be *grounded* if every $\in$-descending chain starting from A

$$\ldots \in x_3 \in x_2 \in x_1 \in A$$

is finite. Let B be the set of all grounded sets. Now, B itself is grounded because if A belongs to B then A is grounded, and as a consequence there is no infinite $\in$-descending chain starting from B. Therefore, $B \in B$ and we can write $\ldots \in B \in B \in B$. Summing up: we proved that B is grounded and, from this very fact, we deduced that B is not grounded.

In [3] it is stressed that, as the diagonal method arises from self-referential paradoxes, also ungrounded arguments can be translated into a proof schema. For instance, to built a non-recursively enumerable set, define a relation E on ω as follows: xEy iff $x \in W_y$, and call an element x grounded if every chain of the kind $\ldots x_3Ex_2Ex_1Ex$ is finite. Then the set $F = \{x/x \text{ is grounded}\}$ is not recursively enumerable. Suppose, on the contrary, that $F = W_z$ for some z: then also z is grounded (since xEz implies that x is grounded); it follows $z \in W_z$, i.e. zEz, and z is ungrounded ($\ldots EzEzEz$). In fact, it is not too hard to prove that F is Π^1_1-complete.

In a similar way we can prove, for instance, two classical Cantor theorems: a function from a set X to its power set $\mathscr{P}(X)$ cannot be surjective, and the set of real numbers is not denumerable (see [3]).

This paper fits into the general program of investigating formal situations in which ungrounded constructions are allowed. In other words, our aim is to find where and how ungrounded sentences can be safely used, keeping in mind that ungrounded sentences occur meaningfully in many informal contexts, from linguistics to computer science, from game theory to probability (see for instance [11, 7]). Moreover, there could be connections with other logical subjects, such as the Anti-Foundation Axiom in set theory and non-well-founded sets (see for instance [1]).

2. First results – a theorem in general topology

The following two propositions give a positive answer to the problem stated at the beginning in two trivial cases.

Proposition. *If X is a finite set, then any function $f: X \to X$ has an unfounded chain (in fact, f must admit a finite cycle).*

Corollary 1. *Let $\mathscr{A}$ be a locally finite algebra (in the sense that every subalgebra generated by a finite set is finite – e.g., a Boolean algebra). Then any polynomial $f: \mathscr{A} \to \mathscr{A}$ has an unfounded chain.*

Proof. It is enough to note that, for every x, f maps the finite subalgebra generated by x and by the parameters that occur in f, into itself. □

Proposition. *If $f: X \to X$ is a surjective function, then f has an unfounded chain.*

Remark. In the proof of the previous proposition (start from any element x_0, choose an x_1 s.t. $fx_1 = x_0$, and so on), we apply a weak form of the axiom of choice, namely the axiom of dependent choices. In the sequel we will often use the axiom of choice in proofs, without indicating it explicitly.

Corollary 2. *A function $f: X \to X$ has an unfounded chain iff there exists a non-empty subset A of X s.t. $A = f(A)$.*

Proof. Given an unfounded chain (a_n), define A to be the set $\{a_n / n \in \omega\} \cup \{f^n a_0 / n \in \omega\}$. The other direction is proved as in the previous proposition. □

We can reformulate somewhat the condition expressed in Corollary 2; for instance: $A \subseteq f(A)$; or $A \subseteq f^n(A)$ for some positive integer n; or for every $x \in A$ there is an n s.t. $x \in f^n(A)$.

A simple necessary condition is the following:

(∗) *If f has an unfounded chain, then $\bigcap_{n\in\omega} Im\, f^n \neq \emptyset$*

(where $Im\, f^n$ denotes the range of the function f^n).

However, the converse of (∗) does not hold: using the language of relations, an element can be founded, without being of finite height (we are thinking of the relation $\mathscr{R}$ defined as follows: $x \mathscr{R} y$ iff $fx = y$).

To get a necessary and sufficient condition, ordinal numbers can be considered. For every $f:X \to X$, define by induction:

$$Im\, f^0 = X; \quad Im\, f^{\alpha+1} = f(Im\, f^\alpha); \quad Im\, f^\alpha = \bigcap_{\delta<\alpha} Im\, f^\delta \quad (\text{if } \alpha \text{ is limit}).$$

Theorem 1. *A function f has an unfounded chain iff all $Im\, f^\alpha$ are non-empty.*

Proof. The statement is a consequence of Corollary 2. If a subset A of X is s.t. $A = f(A)$, then $A \subseteq Im\, f^\alpha$ for every ordinal α. Conversely, if all $Im\, f^\alpha$ are non-empty, since $\alpha < \beta$ implies $Im\, f^\alpha \supseteq Im\, f^\beta$, there is an α s.t. $Im\, f^\alpha = Im\, f^{\alpha+1} = f(Im\, f^\alpha)$. $\square$

Remark. In the statement of Theorem 1, we can obviously restrict ourselves to all ordinal numbers whose cardinality is less than or equal to $\mathrm{Card}\, X$. Similar constructions are familiar in other contexts: in particular, we are referring to the Cantor Bendixson index for topological spaces.

In general topology many fixed-point theorems are known. Obviously, not any compact subset of R^n enjoys the fixed point property: it is enough to consider a disconnected set, or a rotation in a circle S^1.

Theorem 2. *Let X be a compact Hausdorff space and let $f:X \to X$ be a continuous function; then f has an unfounded chain.* [1]

Proof. Using notation as in Theorem 1, we claim that, for every ordinal α, the set $Im\, f^\alpha$ is a non-empty closed subset of X. We proceed by induction. The case $\alpha = 0$ is trivial. If $Im\, f^\alpha$ is closed, it is also compact and, therefore, $Im\, f^{\alpha+1} = f(Im\, f^\alpha)$ is in turn compact; since X is Hausdorff, $Im\, f^{\alpha+1}$ is closed and, obviously, non-empty if so is $Im\, f^\alpha$. If α is a limit ordinal, $Im\, f^\alpha$ is closed, as it is the intersection of closed sets, and is non-empty, since X is compact.

In fact, since X is compact, it would be enough to consider $Im\, f^\alpha$ with $\alpha \leqslant \omega$, in the sense that $Im\, f^\omega = f(Im\, f^\omega)$; see the proof of Theorem 9. $\square$

We get a similar result slightly modifying the hypotheses.

Theorem 3. *Let X be a sequentially compact Hausdorff space and let $f:X \to X$ be a continuous function; then f has an unfounded chain.*

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[1] A problem published in *the American Mathematical Monthly* (October 1995 - n. 10476) asked to prove that *if X is a countable compact Hausdorff space, then every continuous map $f:X \to X$ has a periodic point*. This can be proved defining a binary relation $\mathscr{R}$ on X as follows: $x \mathscr{R} y$ iff y is an accumulation point of the set $\{f^n x / n \in \omega\}$. By Baire's theorem, $\mathscr{R}$ is a strict partial ordering; by Zorn's Lemma, there is a maximal point with respect to $\mathscr{R}$. Lastly, notice that a point x is maximal iff the set $\{f^n x / n \in \omega\}$ is finite iff this set contains a periodic point.

This result is obviously related to the statement of Theorem 2; however, it does not seem to have significant applications in logic, because of the hypothesis that X is countable. In a sense, in this as in other contexts, considering unfounded chains seems to allow simpler statements and more natural hypotheses.

Proof. For any x in X, consider the sequence $(f^n x)$. Let $(f^{n_i}x)$ be a convergent subsequence of it, and let x_0 be its limit. Now, consider the sequence $(f^{n_i-1}x)$ (dropping the first element if it makes no sense): also this has a convergent subsequence $(f^{m_i-1}x)$. If x_1 is its limit, by the continuity of f and the uniqueness of the limit, we have $fx_1 = f(\lim f^{m_i-1}x) = \lim f^{m_i}x = x_0$. Iterating the same procedure, we get the required sequence. □

Example 1. It is readily seen that both hypotheses in Theorems 2 and 3 are needed; for instance, consider the successor function in the set ω with the cofinite topology, which is a compact, sequentially compact non-Hausdorff space.

The next theorem gives a combinatorial characterization.

Theorem 4. *A function f from a non-empty set X into itself has an unfounded chain iff there are two functions $g, h : X \to X$ s.t. $fgh = g$.*

Proof. ($\Leftarrow$) Let $x \in X$. Then $gx = fghx = ffghhx = \cdots$. Defining a_n to be $gh^n x$, we have: $fa_{n+1} = fgh^{n+1}x = gh^n x = a_n$.

($\Rightarrow$) Let (a_n) be a sequence s.t. $fa_{n+1} = a_n$. Take as g any "projection" from X to the set $\{a_n / n \in \omega\}$, i.e. a function s.t. $ga_n = a_n$. Define h as follows: if $gx = a_n$, then $hx = a_{n+1}$. If $gx = a_n$, we have $fghx = fga_{n+1} = fa_{n+1} = a_n = gx$.

Note two particular cases. If g is a constant function, then from $fgh = g$ it follows $fg = g$: therefore f admits a fixed point. If, on the other hand, g is the identity function, then from $fgh = g$ it follows that fh is the identity function: therefore f is surjective. □

Let us close this section by mentioning a question which will not be further discussed in this paper: *when can we find an unfounded chain (a_n) s.t. $a_n \neq a_m$ for $n \neq m$?* It is not hard to prove that this is the case, for instance, of the relation E defined in ω as follows: xEy if $x \in Wy$. To find sufficient conditions, the concept of a *complete* relation (in a denumerable set) can be applied (see [4, 3, 6]).

3. Functions in the set of sentences

We will refer to Peano arithmetic PA, even if what follows applies to other theories as well. Sentences will be identified with their Gödel numbers; for every sentence p, $\bar{p}$ stands for the numeral of (the Gödel number of) p.

Let us recall a classical fixed-point theorem in logic, from which one can deduce the two Gödel Theorems, Tarski Theorem, Löb Theorem, as well as the fact that the set of theorems of PA is effectively inseparable from the set of negations of theorems.

Diagonalization Lemma (DL). *For every formula $H(v)$ of PA, where v is free, there is a sentence p s.t. p is provably equivalent to $H(\bar{p})$.*

Define the relation $\sim_{PA}$ as follows: $x \sim_{PA} y$ if x and y are sentences which are provably equivalent in PA. If sentences are considered up to the relation $\sim_{PA}$, DL is a fixed-point theorem: called Σ the set of sentences of PA, every recursive function from the set Σ into itself, which is induced by a formula $H(v)$, has a fixed-point. However, it is trivial that not every recursive function from Σ into itself has a fixed-point (up to $\sim_{PA}$): it is enough to consider the function induced by the connective $\neg$. On the other hand, this function has obvious cycles of length 2.

A natural conjecture is the following: every recursive function f from the set Σ into itself has an unfounded chain (p_n) up to $\sim_{PA}$ (in the sense that fp_{n+1} is provably equivalent to p_n). But, without further hypotheses, this conjecture is false. Indeed, let (q_n) be an effective sequence of sentences s.t. $q_n \notin \Sigma_n$ for every n. Let f be the function which maps a sentence p in q_n if the normal prenex form of p is a Σ_n-formula (but not a Σ_{n-1}-formula). [2] Obviously, f cannot have an unfounded chain.

In the next section, after setting DL in a general context, we will construct a more interesting counterexample.

Now, let us restrict ourselves to the recursive functions from Σ into itself, that can be expressed as Boolean combinations of:

the identity function,

functions induced by some formula $H(v)$ with the variable v free.

Open problem. *Every such function has an unfounded chain.*

By Corollary 1, f has a finite cycle if all formulas that occur in the construction of f are "constant" ($H(v)$ is "constant" if $H(\bar{p}) \sim_{PA} H(\bar{q})$ for every p and q). The next theorem gives a positive answer in other cases.

Theorem 5. *Let $H(v)$ be a formula. Then each of the following functions has an unfounded chain*:

(i) $fp = p \wedge H(\bar{p})$ *and* $fp = p \vee H(\bar{p})$,

(ii) $fp = p \leftrightarrow H(\bar{p})$,

(iii) $fp = \neg p \wedge H(\bar{p})$ *and* $fp = \neg p \vee H(\bar{p})$.

Proof. (i) These functions have the same fixed-point as $H(v)$.

(ii) The statement follows from the fact that f is surjective (up to $\sim_{PA}$): indeed, given a sentence a, apply DL to the formula $a \leftrightarrow H(v)$ to get a q s.t. $q \sim_{PA} a \leftrightarrow H(\bar{q})$; it is obvious that $a \sim_{PA} q \leftrightarrow H(\bar{q})$.

(iii) Let us prove that the first function has a cycle of length 2; a similar argument applies to the second function. Apply DL to the formula $\neg H(v) \wedge H(\overline{H(v)})$ to get a sentence q s.t. $q \sim_{PA} \neg H(\bar{q}) \wedge H(\overline{H(\bar{q})})$. Then $fq = \neg q \wedge H(\bar{q}) \sim_{PA} H(\bar{q})$, since $q \to \neg H(\bar{q})$ is a theorem. On the other hand, f maps $H(\bar{q})$ in $\neg H(\bar{q}) \wedge H(\overline{H(\bar{q})}) \sim_{PA} q$. $\square$

[2] Throughout the paper, Σ_n denotes the set of sentences which are provably equivalent to a formula with a Σ_n prefix. However here, to obtain a recursive function, we just count the number of quantifiers in the normal prenex form of p.

Remark. In the case (ii) of the previous theorem, we get a sequence (a_n) of sentences, where each a_n "says" that a_{n+1} is provably equivalent to $H(\overline{a_{n+1}})$. For suitable formulas $H(v)$, these sentences can be regarded as ungrounded.

We will see in Section 5, in a more abstract context, that every function built starting from the identity function, Boolean operations, and the function induced by the provability predicate *Theor*(v), has an unfounded chain.

4. A counterexample

To find a general framework for fixed-point theorems, Ershov suggested the concept of a precomplete equivalence relation. Let us briefly review some concepts and results. In the following, we will consider only recursively enumerable (or positive) equivalence relations on the set ω.

Definition 2 (*see for instance Bernardi and Sorbi* [4], *Lachlan* [9] *or Shavrukov* [14]). An equivalence relation $\mathscr{R}$ (different from the trivial relation $\omega \times \omega$) is said to be *precomplete* if for every partial recursive function ψ there is a total recursive function g s.t., if ψ converges on x, then $\psi x \mathscr{R} gx$.

Ershov fixed-point theorem. *Assume that $\mathscr{R}$ is a precomplete equivalence relation. For every total recursive function f there is a fixed-point, that is, a number n s.t. $fn \mathscr{R} n$.*

Proposition (Visser [17]). *Consider Σ_n sentences (more precisely, consider a Gödel numbering just for Σ_n sentences), and define $x \sim_n y$ iff x and y are provably equivalent Σ_n sentences. The relation $\sim_n$ is precomplete.*

Note that the Diagonalization Lemma can be immediately deduced from the quoted results. For, let $H(v)$ be a formula and assume that its prefix is Σ_n. If we map every Σ_n sentence p in $H(\bar{p})$, we have a recursive function from Σ_n into Σ_n. By the Ershov fixed-point theorem, there is a sentence p_0 s.t. $H(\overline{p_0}) \sim_n p_0$, that is, $H(\overline{p_0})$ and p_0 are provably equivalent. On the other hand, $\sim_{PA}$ is not precomplete, because, as we have already noted, there are recursive functions without fixed-points.

ADN Theorem (Visser [17]). *Let $\mathscr{R}$ be a precomplete equivalence relation, and let Δ and ψ be two partial recursive functions; assume that Δ is a diagonal function for $\mathscr{R}$, in the sense that, if $\Delta x\downarrow$, then $\Delta x \not\mathscr{R} x$. Then there is a total recursive function g s.t., if $\psi x\downarrow$, then $\psi x \mathscr{R} gx$, while, if $\psi x\uparrow$, then $gx \notin$ domain Δ. Moreover, an index of g can be found in a uniform way starting from indices of $\Delta, \psi, \mathscr{R}$.*

Coming back to the problem discussed in Section 3, a reasonable hypothesis is expressed by the following.

Definition 3. We say that a function $f : \Sigma \to \Sigma$ is *extensional* if $p \sim_{PA} q$ implies $f \sim_{PA} fq$, for every p and q.

However, we are in a position to built a recursive extensional function h, which admits no unfounded chain up to $\sim_{PA}$. The symbol $[p]$ denotes the set of (Gödel numbers of) sentences which are provably equivalent to p.

Theorem 6. *There exists a total recursive function h, which is extensional and increasing, in the sense that, if $h([p]) \subseteq [q]$, then the minimum of $[p]$ is less than the minimum of $[q]$. Moreover, h is an injective function, up to the relation $\sim_{PA}$.*

Proof. Define h by induction. Let $h0$ be $\neg 0$.

Assuming that $h0, \ldots, hm$ have already been defined, let n be the least number s.t. the sentences $0, \ldots, m, m+1$, $h0, \ldots, hm$ and their negations belong to Σ_n;[3] consider the precomplete relation $\sim_n$. Define two partial recursive functions Δ and ψ as follows:

$\Delta x = \neg x$ for $x \in [0] \cup \cdots \cup [m+1] \cup [h0] \cup \cdots \cup [hm]$ and divergent otherwise (of course, Δ is a diagonal function);

$\psi x = hi$, if $x \sim_n i$ for some $i < m+1$, and divergent otherwise.

Apply the ADN Theorem to get a total recursive function g s.t. $gx \sim_n \psi x$ for all x s.t. $\psi x\downarrow$, and $gx \notin \operatorname{domain} \Delta$ for all x s.t. $\psi x\uparrow$. Define $h(m+1) = g(m+1)$ (strictly speaking, $g(m+1)$ is the Gödel number of a Σ_n sentence when a Gödel numbering just for Σ_n sentences is considered, while $h(m+1)$ is the Gödel number of the same sentence when a Gödel numbering for all sentences is considered).

Now, if $m+1 \sim_{PA} i$ for some $i < m+1$, then $h(m+1) = g(m+1) \sim_n \psi(m+1) = hi$ and therefore $h(m+1) \sim_{PA} hi$ (i.e. h is extensional). On the other hand, if $m+1$ is not provably equivalent to any $i < m+1$, then $m+1 \notin \operatorname{domain} \psi$ and therefore $h(m+1) = g(m+1) \notin \operatorname{domain} \Delta$: thus $h(m+1) \notin [hi]$ (i.e. h is injective), and $h(m+1) \notin [j]$ for all $j \leqslant m+1$ (i.e. h is increasing).

Notice that in any inductive step a particular precomplete equivalence relation is considered (in other words, n depends on m). The resulting function h is recursive because the ADN Theorem holds in a uniform way. □

Corollary 3. *There exists a total recursive extensional function h from the set Σ of sentences of PA into itself, which has no unfounded chain (in particular, h has neither fixed points, nor finite cycles).*

Proof. Obvious. □

Open problem. *Let f be a recursive function s.t. $f(\Sigma_1) \subseteq \Sigma_n$ for some n. Then f has an unfounded chain.*

[3] See footnote 2.

One could add stronger hypotheses about f, as for instance: (1) f is extensional, or (2) $f^r(\Sigma_1) \subseteq \Sigma_n$ for all r. Note that Boolean combinations considered in the previous section satisfy the second hypothesis.

5. A theorem in Magari algebras

Let us recall that a *Magari algebra* (or a *diagonalizable algebra* – briefly an *MA*) is a Boolean algebra $\langle B; 0, 1, \vee, \wedge, \neg \rangle$ endowed with a unary operator $\Box$, s.t. the following identities hold: $\Box 1 = 1$, $\Box(x \wedge y) = \Box x \wedge \Box y$, $\Box(x \vee \neg\Box x) = \Box x$.

The main example of an *MA* is the Lindenbaum algebra of *PA*: $\Box$ is defined as $\Box[p] = [Theor(\bar{p})]$, where $Theor(v)$ is the standard provability predicate. *MA's*, as well as the corresponding modal logic *GL*, provide a suitable algebraic framework to study incompleteness phenomena – see for instance [5, 16], or [13] for more recent results. In particular, the Diagonalization Lemma admits the following translation:

Fixed-point theorem (see Bernardi [2], Sambin [12]). *Let h be a polynomial in the variable x s.t. x occurs only within the scope of a $\Box$. Then in every MA there exists an element a s.t. $ha = a$.*

Several proofs of this theorem are known. Moreover, it has been generalized in various respects; in 1993 Mardaev [10] dropped the hypothesis that the variable x occurs only within the scope of a $\Box$, and found fixed points for polynomials in which all occurrences of x are positive. However, it is obvious that there are terms [4] (e.g. $\neg x$) with no fixed point. In the next theorem, we intend to show that in *MA's* unfounded chains can be constructed, with respect to a larger and more natural class of functions.

Theorem 7. *Let f be a term in one variable. Then in every MA there exists a sequence (a_n) s.t. $f a_{n+1} = a_n$ for all n.*

The proof will be divided into four lemmas.

First remarks and notation. Since only constants are involved, the statement holds in every *MA* iff it holds in the free *MA* $\mathscr{F}_0$ on the empty set. From a logical point of view, the elements of $\mathscr{F}_0$ can be obtained starting from 1 (set of theorems) and 0 (set of refutable sentences), and applying Boolean operations and the operator *Con*, where $Con[p]$, for every sentence p, is the consistency of the theory $PA + p$.

As is proved in [2], the Boolean structure of $\mathscr{F}_0$ is isomorphic with the algebra of finite and cofinite subsets of ω; so $\mathscr{F}_0$ can be regarded as a Boolean subalgebra of $\mathscr{P}(\omega)$. Moreover, the operator $\Box$ can be extended to $\mathscr{P}(\omega)$ considering "$>$" in ω

[4] Speaking of a *term* in an algebra $\langle A; F \rangle$, where F is a set of operations defined in the set A, we mean a function from A^n into A built from the elements of F. Replacing the occurrences of some variables by elements of A, we get a *polynomial*.

as the accessibility relation in the usual modal meaning; in this way, $\mathscr{F}_0$ becomes a Magari subalgebra of $\mathscr{P}(\omega)$, and the term f can be extended to $\mathscr{P}(\omega)$.

We identify $\mathscr{P}(\omega)$ with the Cantor set $\mathbf{2}^\omega$, i.e. with a subset of the real line with the usual topology; it is not difficult to check that $\mathscr{F}_0$ consists of all sequences which are definitely constant, or, in other words, of the extremes of all removed intervals. If $x \in \mathbf{2}^\omega$, the elements of the sequence x will be called *digits* of x (we consider the digits 0 and 1, even if in the Cantor set the digits 0 and 2 are usually considered). Note that, when $x, y \in \mathscr{F}_0$ are regarded as real numbers, $\neg x$ is $1 - x$, whereas $x \wedge y$ and $x \vee y$ are not the minimum and the maximum between x and y.

If $x \in \mathbf{2}^\omega$ and $A \subseteq \omega$, we write $x|_A$ to denote the restriction of x to A; in particular, assuming as usual that $n + 1 = \{0, \dots, n\}$, the symbol $x|_n$ stands for the first n digits of x. Let U_x^n be the set $\{y \in \mathbf{2}^\omega / y|_n = x|_n\}$, i.e. the neighborhood of x determined by the first n digits of x; of course, every U_x^n is a clopen set.

Since f is a continuous function and the Cantor set $\mathbf{2}^\omega$ is compact, we can apply Theorem 2 to find an unfounded chain in $\mathbf{2}^\omega$: the point is to show that an unfounded chain exists in $\mathscr{F}_0$.

Example 2. There exists a continuous function k from $\mathbf{2}^\omega$ into itself, s.t. $k(\mathscr{F}_0) \subseteq \mathscr{F}_0$ and no unfounded chain exists in $\mathscr{F}_0$. Indeed, define $kx = \langle 1 \rangle * x$ if the first digit of x is 0 and $kx = \langle 0 \rangle * x$ if the first digit of x is 1 (where $*$ stands for concatenation).

Lemma 1. (i) *Let f be a term. For every $x \in \mathbf{2}^\omega$ and every n, the first n digits of fx depend only on the first n digits of x.*

(ii) *For every x there exists an n s.t., in the set $\omega - n$, the value fx coincides either with* 0, *or with* 1, *or with x, or with $\neg x$.*

Proof. Trivial, by induction on the structure of f. $\square$

Lemma 2. *As regards the four possibilities in Lemma* 1(ii), *the behaviour of fx can depend on x,*[5] *but, if $x \in \mathbf{2}^\omega - \mathscr{F}_0$, it is the same for all points in some neighborhood of x. More precisely*,

(i) *if $x \notin \mathscr{F}_0$ and $fx \in \mathscr{F}_0$, then there is an n s.t. $fy = fx$ for all y in U_x^n (f is locally constant)*;

(ii) *if both x and fx belong to $\mathbf{2}^\omega - \mathscr{F}_0$, then there is an n s.t. either for all y in U_x^n we have $fy|_n = fx|_n$ and $fy|_{\omega - n} = y|_{\omega - n}$, or for all y in U_x^n we have $fy|_n = fx|_n$ and $fy|_{\omega - n} = \neg\, y|_{\omega - n}$; as a consequence, there is an n s.t. $f(U_x^n) = U_{fx}^n$ and, from a geometrical point of view, f is locally an isometry.*

[5] For example, let fx be the term $[x \wedge \Box(x \vee \neg\Box 0)] \vee [\neg x \wedge \Box(x \vee \Box 0 \vee \neg\Box^2 0)]$ and think of its domain as $\mathscr{P}(\omega)$ instead of $\mathbf{2}^\omega$: if x is the set of even numbers, fx eventually coincides with x; if x is the set of odd numbers, fx eventually coincides with $\neg x$; if x is the set P of positive even numbers, fx eventually coincides with 0; if x is the set $\omega - P$, fx eventually coincides with 1.

Proof. Also this proof is by induction on the structure of f. It is obvious that the term x and the constant term 0 satisfy the statement; and it is readily seen that, if two terms g and h satisfy the statement, so do $\neg g$ and $g \wedge h$. Then consider a term g of the form $\Box h$ for some term h (note that, if this is the case, $gx \in \mathscr{F}_0$); there are two possibilities: either $\Box hx = 1$, or $\Box hx \neq 1$. In the first case, $hx = 1$ and, by the inductive hypothesis, h takes the value 1 in all the points of a neighborhood of x (note that the equalities $\Box x = 1$ and $\Box x = x$ do not hold in any neighborhood of $x = 1$; therefore the statement does not hold if $x \in \mathscr{F}_0$). In the second case, called m the least number s.t. the mth digit of hx is 0, by Lemma 1(i) the term $\Box h$ takes the same value (that is, $\Box^{m+1}0$) for all $y \in U_x^{m+1}$.

Now, keeping in mind the proofs of Theorems 1 and 2, let A be the subset of $\mathbf{2}^\omega$ defined as $A = \bigcap f^\alpha \mathbf{2}^\omega$ where the intersection is taken over the class of all ordinals α; we know that $f(A) = A$ and that A is (non-empty and) compact. $\Box$

Lemma 3. *If $A \cap \mathscr{F}_0 \neq \emptyset$, then there is an unfounded chain (a_n) in $\mathscr{F}_0$.*

Proof. Let a_0 be an element of $A \cap \mathscr{F}_0$. Since $a_0 \in A$, by Corollary 2 there is a chain (a_n) in A s.t. $fa_{n+1} = a_n$: it is enough to prove that also a_1 belongs to $\mathscr{F}_0$ because the same argument can be iterated. We claim that, since $a_1 \in A$ and $fa_1 \in \mathscr{F}_0$, also a_1 must belong to $\mathscr{F}_0$. Assume $a_1 \notin \mathscr{F}_0$. Since $fa_1 \in \mathscr{F}_0$ we can apply Lemma 2(i): restrict all elements in the chain to the first n digits, where n is as in Lemma 2(i) for fa_1. There must exist two numbers i, j $(i > j > 1)$ s.t. $a_i|_n = a_j|_n$; by Lemma 1(i) we have $f^{j-1}a_i|_n = f^{j-1}a_j|_n = a_1|_n$. From $fa_1 = a_0 \in \mathscr{F}_0$, by Lemma 2(i), it follows $f(f^{j-1}a_i) = a_{i-j} \in \mathscr{F}_0$; we conclude that $a_1 = f^{i-j-1}a_{i-j} \in \mathscr{F}_0$. $\Box$

The proof of Theorem 7 is concluded by

Lemma 4. $A \cap \mathscr{F}_0 \neq \emptyset$.

Proof. We proceed by absurd; more precisely the schema of the proof is the *consequentia mirabilis*: assuming $A \cap \mathscr{F}_0 = \emptyset$, we prove $A \cap \mathscr{F}_0 \neq \emptyset$.

For every x in A, find a neighborhood satisfying the conditions of Lemma 2(ii): we have a covering of A, which, since A is compact, admits a finite subcovering $(U_x)_{x \in K}$, K being a finite subset of A. We can also assume that all U_x are associated with the same value of n (in geometrical words, all U_x have the same length), so that we can write $(U_x^n)_{x \in K}$. By Lemma 2(ii), we know that the term f maps each U_x^n onto U_{fx}^n: so U_{fx}^n must be equal to U_y^n for some $y \in K$ (indeed, fx belongs to A and therefore to some U_y^n). In other words, f induces a bijection from the set $\{U_x^n / x \in K\}$ into itself, and $f(\bigcup U_x^n) = \bigcup U_x^n$: it follows that $\bigcup U_x^n = A$ (a priori, we only knew that $\bigcup U_x^n \supseteq A$). But every U_x^n contains elements of $\mathscr{F}_0$, which, therefore, belong to A. $\Box$

Open problem. *The statement of Theorem 7 holds also in the case f is a polynomial in one variable.* (Note that Lemmas 1(ii) and 2 do not hold for polynomials in the MA $\mathscr{P}(\omega)$.) *Moreover, if* $f(x, y_1, \ldots, y_k)$ *is a polynomial in* $k+1$ *variables, then there is a sequence* (a_n) *of polynomials in the variables* $y_1, \ldots, y_k$ *s.t.* $f(a_{n+1}(y_1, \ldots, y_k), y_1, \ldots, y_k) = a_n(y_1, \ldots, y_k)$.

6. An example

In this section we will discuss an example of a typical unfounded chain generated by a term. For the sake of brevity, we will omit the proofs of some statements.

Let gx be the term $x \leftrightarrow \neg\Box x$, that is, $gx = \neg x \leftrightarrow \Box x$, or also $gx = \neg(x \leftrightarrow \Box x)$. In every MA:

(i) gx admits no fixed point: from $a = a \leftrightarrow \neg\Box a$ it follows $\Box a = 0$;

(ii) $gx = y$ iff $g(\neg y) = \neg x$, and therefore $(\neg g)^2$ is the identity function;

(iii) from the previous point it follows that g is a bijection;

(iv) for every term h in which x occurs only within the scope of a $\Box$, there is one and only one element a s.t. $ha = ga$: indeed, $hx = x \leftrightarrow \neg\Box x$ is equivalent to $x = hx \leftrightarrow \neg\Box x$, and the latter equation admits just one solution because any term in which x occurs only within the scope of a $\Box$ has exactly one fixed point.

In $\mathscr{F}_0$ the term g does not admit finite cycles, but yields a unique chain (of order type Z):

$$\cdots \mapsto \neg\Box 0 \mapsto 1 \mapsto 0 \mapsto \Box 0 \mapsto \Box^2 0 \wedge \neg\Box 0 \mapsto \Box^2 0 \mapsto \cdots .$$

In fact, g enjoys a nice combinatorial property. Let $x \in \mathscr{F}_0 \subseteq \mathbf{2}^\omega$ and assume first that the digits of x are definitely equal to 0; read the sequence x from right to left as a natural number in binary notation (for instance, $\Box^2 0 \wedge \neg\Box 0$ is the sequence $(0, 1, 0, 0, 0, \ldots)$, that represents the number 2). One can prove that, under this coding, $gn = n + 1$. Similarly, if the digits of x are definitely equal to 1, to compute gx it is enough to "add 1" to the sequence x, again starting with the first digit at the left and taking into account the fact that in each place $1 + 1$ gives 10, the 1 of which must then be carried over to the sum at the next place on the right. In particular, the Boolean element 1 is identified with the sequence with all digits equal to 1: adding 1, we get the sequence with all digits equal to 0, i.e. $g1 = 0$.

It is nearly trivial that, in $\mathscr{F}_0$, the term g^n yields n chains of type Z.

In some infinite MA's the term gx admits finite cycles: for instance, it is known that in the MA of PA there is an element p s.t. $\Box p = \Box\neg p$ (think of a Rosser sentence). In this case, one can prove that $g^2 p = p$, i.e. there is a cycle of length 2.

From a logical point of view, identifying sentences with their equivalence classes, gx is a true sentence iff so is x, with the only exception of $g1 = 0$; so, the theory $PA + g^{-1}x$ is sound iff so is the theory $PA + x$, unless x is a contradiction. Moreover, $g^{-1}x$ is not a theorem of the theory $PA + x$ (if this theory is consistent), and is a refutable sentence of the same theory iff $x \to \Box x$ is a theorem of PA.

7. A conjecture in recursion theory and a more general problem

Open problem. *Let f be a total recursive function s.t., if ϕ_x is a characteristic function, so is ϕ_{fx}. Then f has an unfounded chain in the set of indices of characteristic functions, in the sense that there is a sequence (x_n) of numbers s.t., for every n, x_n and fx_{n+1} are indices of the same characteristic function.*

This statement is connected to the recursion theorem: the latter refers to fixed points for r.e. sets, while the former to unfounded chains in the case of decidable sets. Hypotheses about f could be weakened, considering a partial recursive function which converges on indices of characteristic functions, or strengthened, considering only extensional functions (i.e. functions s.t. if x and y are indices for the same characteristic function, also fx and fy are indices for the same characteristic function).

In fact, it is not completely trivial to construct an example of an extensional function without finite cycles. To this end, represent a subset of ω as a sequence in $\{0,1\}$, and write it from right to left (for instance, the singleton of 2 becomes the sequence "$\ldots 00100$", with just one digit equal to 1). Let f be the function that maps a subset D of ω in the set $D+1$, where the sum has to be carried out in binary notation (for instance, $\ldots 10101 + 1 = \ldots 10110$; in particular, we assume $\ldots 111 + 1 = \ldots 000$): f has no finite cycles, but has unfounded chains.

If f can be extended to the set $\mathscr{P}(\omega) \cong \mathbf{2}^\omega$, the situation looks like that discussed in Section 5: since f is a continuous function and the Cantor set $\mathbf{2}^\omega$ is compact, we can apply Theorem 2 to find an unfounded chain in $\mathbf{2}^\omega$: the point is to show that an unfounded chain exists in the set of indices of decidable sets.

The problem introduced at the beginning of this paper can be generalized to a sequence of functions, according to the following.

Definition 4. Let (f_n) be a sequence of functions from a set X into itself. We say that a sequence (x_n) of elements of X is an *unfounded chain* for (f_n) if $f_n x_{n+1} = x_n$ for every n.

Many results proved in Section 2 can be generalized: even if the proofs sometimes become slightly more technical, it does not make a great difference if we consider a sequence of functions instead of a single function. Let us see some results in the new context.

Theorem 8. *If X is a finite set, then any sequence of functions $f_n : X \to X$ admits an unfounded chain.*

Proof. Define a tree T in the following way: the set of nodes is the set $X \times \omega \cup \{\alpha\}$, where $\alpha \notin X \times \omega$, while the edges of T join every node of the kind $(x;\, n+1)$ with $(fx;\, n)$, and $(x;\, 0)$ with α (α being the top of T). Now, the tree T is infinite and each node has finitely many neighbors: so, by König's infinity Lemma, T has an infinite branch, which corresponds to an unfounded chain. $\square$

Proposition. If (f_n) is a sequence of surjective functions, then (f_n) admits an unfounded chain.

Theorem 9 (see Theorem 2). *Let X be a compact Hausdorff space and let $f_n : X \to X$ be a sequence of continuous functions; then (f_n) has an unfounded chain.*

Proof. Consider the sets $X, f_0X, f_0f_1X, f_0f_1f_2X, \ldots$, and let A_0 be their intersection. Similarly, starting from the sets $X, f_iX, f_if_{i+1}X, \ldots$, call A_i their intersection. As in the proof of Theorem 2, all these sets are closed and non-empty; notice that induction over ordinal numbers is unnecessary. We claim that the equalities $f_0A_1 = A_0, f_1A_2 = A_1, \ldots$ hold, from which the statement follows. The inclusion $f_0A_1 \subseteq A_0$ is obvious; on the other hand, if $a_0 \in A_0$, then, for each n, $a_0 \in f_0f_1 \ldots f_nX$; hence, there is a $y_n \in f_1 \ldots f_nX$ s.t. $f_0y_n = a_0$. Call a_1 an accumulation point of the sequence (y_n); keeping in mind that A_1 is closed and f_0 is continuous, we conclude that $a_1 \in A_1$ and $f_0a_1 = a_0$. Therefore, $A_0 \subseteq f_0A_1$. In the same way, the other equalities are proved. □

Theorem 10 (see Theorem 3). *Let X be a sequentially compact Hausdorff space and let $f_n : X \to X$ be a sequence of continuous functions; then (f_n) has an unfounded chain.*

Proof. For any x in X, consider the sequence $f_0x, f_0f_1x, f_0f_1f_2x, \ldots$. Let (y_n) be a convergent subsequence of it, and let x_0 be its limit. Now, "drop the f_0" at the beginning of all y_n: also this sequence has a convergent subsequence (z_n). If x_1 is its limit, by the continuity of f_0 and the uniqueness of limit, we have $f_0x_1 = f_0(\lim z_n) = \lim f_0z_n = \lim y_n = x_0$. Iterating the same procedure (and removing the first element whenever it makes no sense), we get the required sequence. □

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